A Two-Person Game for Pricing Convertible Bonds

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What is a convertible bond?

It is a bond issued by a firm with the following provisions:

- **Pays coupons.**
  In our model, coupons are paid continuously at rate $c > 0$.

- **Can be called by the firm.**
  In our model, the firm may call at any time the firm value exceeds $K$ by offering to pay $K$. $K > 0$ is the *call price*.

- **Can be converted to stock by the bondholder.**
  In our model, the bondholder may convert the bond to stock worth a fraction $\gamma \in (0, 1)$ of the value of the firm. $\gamma$ is the *conversion factor*.

- **Rules of the call.**
  If the firm calls, the bondholder may *surrender* the bond in exchange for payment $K$ or may *convert* it to stock worth a fraction $\gamma$ of the value of the firm.

**Two-person zero-sum game.** The firm seeks a call strategy that *minimizes* the value of the bond. The bondholder seeks a conversion strategy that *maximizes* its value.
Model assumptions.

- **Volatility** – Value of firm has constant volatility $\sigma > 0$.
- **Interest rate** – Constant rate of interest $r > 0$.
- **Dividends** – Firm continuously pays a fixed fraction $\delta \in (0, 1)$ of its equity value as dividends.

Standing Assumption. $\delta < r$.

Notation.

- Value of firm – $X(t)$
- Value of convertible bond – $D(t)$
- Equity value of firm – $S(t)$
- Miller-Modigliani – $X(t) = D(t) + S(t)$

Dynamics prior to call, conversion and bankruptcy

$$dX(t) = rX(t)\,dt + \sigma X(t)\,dW(t) - c\,dt - \delta S(t)\,dt$$
Literature.


- Values convertible bond as a contingent claim on the firm value.
- Dividends and coupons are paid at discrete dates.
- Conversion and/or call occurs only immediately prior to dividend payments. Between dividend payments, firm value evolves as a geometric Brownian motion.
- Optimal call and conversion policies are found by a backward recursion over payment dates.

- Values convertible bond as a contingent claim on the firm value.
- Does not consider the possibility of default, except at maturity.
- Argues that when no dividends are paid, optimal conversion does not occur until maturity and call should occur the first time $\gamma X(t) = K$.
- Some qualitative results are stated when dividends are positive and a function of the firm value.
- Companion paper observes that firms seem to delay call and presents some possible reasons for this.
Dynamics prior to call, conversion and bankruptcy.

\[ dX(t) = rX(t) \, dt + \sigma X(t) \, dW(t) - c \, dt - \delta (X(t) - D(t)) \, dt. \]

We seek a function \( g(t, x) \) such that \( D(t) = g(t, X(t)) \).

**Properties of \( g \).**

1. \( g(t, x) \geq \gamma x \)
2. \( g(t, x) = \gamma x \) for \( x \geq \frac{K}{\gamma} \)
3. \( g(t, x) \leq K \) for \( x \leq \frac{K}{\gamma} \)
4. \( 0 \leq g(t, x) - g(t, y) \leq y - x \) for \( y \geq x \).

Let \( \mathcal{G} \) be the set of continuous functions on \([0, T] \times [0, \infty)\) satisfying (1)–(4).
Given $g \in \mathcal{G}$, $s \in [0, T]$ and $x \geq 0$, define $X^{s,x}$ by $X^{s,x}(s) = x$ and

$$dX^{s,x}(t) = rX^{s,x}(t)dt + \sigma X^{s,x}(t)dW(t) - cdX^{s,x}(t) - \delta \left[ X^{s,x}(t) - g(t, X^{s,x}(t)) \right] dt.$$ 

**Time of bankruptcy:**

$$\tau^{s,x}_0 \triangleq \inf \{ t \in [s, T] : X^{s,x}(t) = 0 \}.$$ 

**Time of conversion:**

$$\tau \in \mathcal{S}^{s,x} \triangleq \{ \text{Stopping times } \theta \in [s, T \wedge \tau^{s,x}_0] \cup \{ \infty \} \}.$$ 

**Time of call:**

$$\rho \in \mathcal{S}^{s,x}_K \triangleq \{ \theta \in \mathcal{S}^{s,x} \text{ such that } X^{s,x}(\theta) \geq K \text{ if } \theta < \tau^{s,x}_0 \}.$$ 

**Risk-neutral expected payoff of the game:**

$$J_g(s, x, \rho, \tau) \triangleq \mathbb{E} \left[ \int_s^{\rho \wedge \tau \wedge T} e^{-r(u-s)}c\,du + e^{-r(\rho \wedge \tau \wedge T-s)} \left( I_{\{\tau \leq \rho \wedge T\}} \gamma X^{s,x}(\tau) + I_{\{\rho < \tau\}} K + I_{\{\rho = \infty, \tau = \infty\}} (L \wedge X^{s,x}(T)) \right) \right].$$ 

**Par value:** $L \leq K$
Lower value of the game:

\[ \underline{v}_g(s, x) \triangleq \sup_{\tau \in S^{s,x}} \inf_{\rho \in S^{s,x}_K} J_g(s, x, \rho, \tau). \]

Upper value of the game:

\[ \overline{v}_g(s, x) \triangleq \inf_{\rho \in S^{s,x}_K} \sup_{\tau \in S^{s,x}} J_g(s, x, \rho, \tau). \]

**Theorem 1 (Value of the game).** The game corresponding to \( g \) has a value, i.e.,

\[ \underline{v}_g = \overline{v}_g. \]

We define \( v_g \) to be this common value.
Value of $v_g$ at maturity.

Define

$$f(x) \triangleq \begin{cases} 
  x & \text{if } 0 \leq x \leq L, \\
  L & \text{if } L \leq x \leq \frac{L}{\gamma}, \\
  \gamma x & \text{if } x \geq \frac{L}{\gamma}.
\end{cases}$$
Value of $v_g$ on boundaries.

\[ x = \frac{K}{\gamma} \]

\[ v_g = K \]

\[ t = 0 \]

\[ v_g = 0 \]

\[ t = T \]

\[ v_g = f \]
Characterization of $v_g$.

**Case I:** $r \geq \frac{c}{K}$. Conversion precedes call and $v_g$ is the unique continuous viscosity solution of

$$
\min \left\{ - v_t + rv - (rx - c)v_x + \delta(x - g)v_x - \frac{1}{2}\sigma^2 x^2 v_{xx} - c, \; v - \gamma x \right\} = 0
$$

satisfying the boundary conditions

$$v(t, 0) = 0, \; v\left(t, \frac{K}{\gamma}\right) = K \text{ for } 0 \leq t \leq T, \; v(T, x) = f(x) \text{ for } 0 \leq x \leq \frac{K}{\gamma}.\tag{2}$$

**Idea of proof.** $h(t, x) = K$ is a supersolution of (1), so $h$ dominates the solution $v$ of this equation.
Characterization of $v_g$ (continued).

**Case II:** $\delta \leq \frac{c}{K}$. Call precedes conversion and $v_g$ is the unique continuous viscosity solution of

$$\max \left\{ -v_t + rv - (rx - c)v_x + \delta (x-g)v_x - \frac{1}{2} \sigma^2 x^2 v_{xx} - c, \ v - K \right\} = 0$$

satisfying the boundary conditions (2).

**Idea of proof.** $h(t, x) = \gamma x$ is a subsolution of (3) so $h$ is dominated by the solution $v$ of this equation.
Overlap of Case I and Case II: $\delta \leq \frac{c}{K} \leq r$.

Call and conversion coincide when firm value reaches $\frac{K}{\gamma}$, and $v_g$ is the unique viscosity solution of

$$-v_t + rv - (rx - c)v_x + \delta(x - g)v_x - \frac{1}{2}\sigma^2 x^2 v_{xx} = c$$

(4)

satisfying the boundary conditions (2).

**Remark.** We have fixed an arbitrary $g \in \mathcal{G}$. The function $v_g$ is the viscosity solution of (4) subject to (2). We have not assumed Hölder continuity of $g$, so we do not know that $v_g$ is smooth enough to be a classical solution of (4).
Theorem 2 (Fixed point). Let $g_1$ and $g_2$ be in $G$, and let $v_{g_1}$ and $v_{g_2}$ be as described above. Then $v_{g_1}$ and $v_{g_2}$ are in $G$ and

\[ \sup_{t,x} |v_{g_1}(t, x) - v_{g_2}(t, x)| \leq \frac{\delta}{r} \sup_{t,x} |g_1(t, x) - g_2(t, x)|. \]

In particular, there exists a unique function $g \in G$ such that $v_g = g$.

Idea of proof (Jensen and Ishii):

Use viscosity solution arguments to bound the function

\[ (t, x, y) \mapsto v_{g_1}(t, x) - v_{g_2}(t, y) - \frac{\alpha}{2} |x - y|^2, \]

and then let $\alpha \to \infty$.

Remark. Let $v^*$ be the fixed point of Theorem 2. Prior to call, conversion and bankruptcy, the price of the bond is

\[ D(t) = v^*(t, X(t)), \]

where

\[ dX(t) = r X(t) \, dt + \sigma X(t) \, dW(t) - c \, dt - \delta \left[ X(t) - v^*(t, X(t)) \right] \, dt. \]
Theorem 3 (Characterization of bond price).

**Case I:** $r \geq \frac{c}{K}$. Conversion precedes call and $v^*$ is the unique continuous viscosity solution of the equation

$$
\min \left\{ -v_t + rv - (rx - c)v_x + \delta(x - v)v_x - \frac{1}{2} \sigma^2 x^2 v_{xx} - c, \quad v - \gamma x \right\} = 0
$$

satisfying the boundary conditions (2).

**Case II:** $\delta \leq \frac{c}{K}$. Call precedes conversion and $v^*$ is the unique continuous viscosity solution of the equation

$$
\max \left\{ -v_t +rv - (rx - c)v_x + \delta(x - v)v_x - \frac{1}{2} \sigma^2 x^2 v_{xx} - c, \quad v - K \right\} = 0
$$

satisfying the boundary conditions (2).

**Overlap of Case I and Case II:** $\delta \leq \frac{c}{K} \leq r$.

Call and conversion coincide when firm value reaches $\frac{K}{\gamma}$, and $v^*$ is the unique viscosity solution of the equation

$$
-v_t + rv - (rx - c)v_x + \delta(x - v)v_x - \frac{1}{2} \sigma^2 x^2 v_{xx} = c
$$

satisfying the boundary conditions (2).
Asymptotic behavior.

For fixed maturity $T$, let $v^*_T(t, x), 0 \leq t \leq T$, denote the price of the bond at time $t$ if $x$ is the firm value. This price depends only on the time to maturity $\tau = T - t$, i.e., there is a function $f(\tau, x)$ such that

$$f(\tau, x) = v^*_T(t, x).$$

**Theorem 4.** The limit

$$f(x) = \lim_{\tau \to \infty} f(\tau, x), \quad x \geq 0,$$

exists and is the price of the *perpetual convertible bond*.

**Idea of proof.** One shows that the convergence in Theorem 4 is uniform in $x$ and that the limiting function $f(x)$ is the unique continuous viscosity solution of the autonomous versions of the differential equations for the two cases in Theorem 3.

These equations were shown in

**M. Sîrbu, I. Pikovsky and S. Shreve,** *Perpetual convertible bonds*, SIAM J. Control Optim. to appear,

to characterize the perpetual convertible bond price.
Asymptotic behavior (continued).

Case I: $r \geq \frac{c}{K}$.

Large call price.

When $r \geq c/K$, the game reduces to the optimal stopping problem of *optimal conversion*. Call occurs at firm value $K/\gamma$, which is greater than or equal to the optimal conversion level.
Asymptotic behavior (continued).

Case II: \( \delta \leq \frac{c}{K} \).

Small call price.

When \( \delta \leq c/K \), the game reduces to the optimal stopping problem of optimal call. Conversion occurs at firm value \( K/\gamma \), which is greater than or equal to the optimal call level.
Asymptotic behavior (continued).

Overlap of Case I and Case II: \( \delta \leq \frac{c}{K} \leq r \).

Intermediate call price.

When \( \delta \leq \frac{c}{K} \leq r \), \textit{call and conversion occur simultaneously} at firm value \( K/\gamma \).
Additional Literature


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